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A Central Limit Theorem for isotropic flows

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Abstract

We establish that the image of a measure, which satisfies a certain energy condition, moving under a standard isotropic Brownian flow will, when properly scaled, have an asymptotically normal distribution under almost every realization of the flow. We derive the same result for an initial point mass moved by an isotropic Kraichnan flow.
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1. Introduction

A general problem in fluid dynamics is to describe the behavior of a body of passive tracers being carried by a random current or turbulent action of a fluid. Real world examples of a body of passive tracers would be an oil slick on the surface of the ocean or a mass of plankton, both of which would be dispersed in some way by the random current of the ocean. Since turbulent actions are approximated by stochastic flows, a reasonable approach is to consider how a body of passive tracers behaves under stochastic flows. As is usually the case, the advantage of the model using stochastic flows over that of random currents lies in the availability of the powerful tools of stochastic differential equations. The objective of this paper is to establish an $a.s.$ Central Limit Theorem for the asymptotic distribution of a body of passive tracers carried by some canonical

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stochastic flows. In more precise terms, we establish an $a.s.$ Central Limit Theorem for the flow of transition probabilities defined by the Kraichnan model of isotropic flows as well as for the more 'traditional' isotropic Brownian flows on $\mathbf{R^d}$. See [3,4,6,1] and [7] for related work on these flows.

We describe now the results of [3] which inspired our interest in this problem. Consider smooth, divergence free vector fields $\{V_k : k = 0, 1, 2, \ldots, m\}$ on a smooth compact manifold M. Let $B = (B_1, B_2, \ldots, B_m)$ be an m-dimensional Brownian motion. Some additional hypoellipticity and nondegeneracy conditions need to be imposed here but we shall not go into the details. Consider the flow generated by the SDE

$$\begin{aligned} \mathrm{d}\phi_t(x) &= \sum_{k=1}^{m} V_k(\phi_t(x))\partial B_k(t) + V_0(\phi_t(x))\mathrm{d}t, \\ \phi_0(x) &= x. \end{aligned} \tag{1}$$

Let A_t be a centered additive functional of this flow. Finally take a measure ν on M with finite energy as measured by a singular kernel similar to the Green function. If

$$\nu_t(K) = \nu\left\{x \in M : \frac{A_t}{\sqrt{t}} \in K\right\}$$

then ν_t converges weakly to a nondegenerate centered Gaussian measure. We consider a counterpart for a measure on passive tracers carried by measure preserving isotropic Brownian and measure preserving isotropic Kraichnan flows on $\mathbf{R^d}$. In these cases, we will show that the analogous result can be proved rapidly. After our work was completed we learned of a recent result of Dimitroff and Scheutzow [2], in which they establish weak convergence to the normal distribution in probability, as opposed to almost surely. Their result requires weaker assumption on the initial measure.

2. Kraichnan flows

Let us first recall some basic features of measure preserving (Lebesgue measure) isotropic Brownian flows. The starting point is an isotropic covariance function $\mathbf{C}^{ij}(x, y) = \mathbf{C}^{ij}(x - y)$ for $x, y \in \mathbf{R^d}$, $i, j \in \{1, 2, \ldots, d\}$. Isotropy in this context means that for any $U \in \mathcal{O}(d)$ and $z \in \mathbf{R^d}$,

$$U^i_k U^j_l \mathbf{C}^{kl}(z) = \mathbf{C}^{ij}(Uz).$$

The structure of the covariance functions associated to measure preserving isotropic flows has been known at least since [8]. Following the development in [6], one has for $z \in \mathbf{R^d}$

$$\mathbf{C}^{ij}(z) = \delta^{ij} B_N(\|z\|) + \frac{z^i z^j}{\|z\|^2}\left(B_L(\|z\|) - B_N(\|z\|)\right). \tag{2}$$

The functions B_L and B_N depend on a spectral measure F

$$B_L(r) = \int_0^\infty \int_{S^{d-1}} \cos(r\rho u_1)\omega(\mathrm{d}u)F(\mathrm{d}\rho) - \int_0^\infty \int_{S^{d-1}} \cos(r\rho u_1)u_1^2\omega(\mathrm{d}u)F(\mathrm{d}\rho) \tag{3}$$

and

$$B_N(r) = \int_0^\infty \int_{S^{d-1}} \cos(r\rho u_1)\omega(\mathrm{d}u)F(\mathrm{d}\rho) - \int_0^\infty \int_{S^{d-1}} \cos(r\rho u_1)u_2^2\omega(\mathrm{d}u)F(\mathrm{d}\rho), \tag{4}$$

where ω is normalized Lebesgue measure on S^{d-1} and F is a finite positive measure on $\mathbf{R}^+$. Set $B = B_L(0) = B_N(0) = 1$ for the sake of simplification. A covariance may be viewed as a map $\mathbf{C} : T^*\mathbf{R^{2d}} \to \mathbf{R}$ such that for any $\xi_1, \xi_2, \ldots, \xi_n \in T^*\mathbf{R^d}$ it holds that

$$\sum_{i,j} \mathbf{C}(\xi_i, \xi_j) \geq 0.$$

For any $\xi = (x, u)$, the covariance $\mathbf{C}$ generates a vector field $\mathbf{C}_\xi$ which is specified by

$$\left\langle \mathbf{C}_\xi(y), v \right\rangle = \mathbf{C}(\xi, \eta),$$

where $\eta = (y, v) \in T^*\mathbf{R^d}$. Take $\mathcal{H}_0$ to be the vector space generated by the vector fields $\mathbf{C}_\xi$ endowed with the inner product

$$\left\langle \mathbf{C}_\xi, \mathbf{C}_\eta \right\rangle = \mathbf{C}(\xi, \eta)$$

and associated norm $\|\ \|$. Denote by $\mathcal{H}$, which is in fact a space of continuous vector fields, the separable completion of $\mathcal{H}_0$ with respect to $\|\ \|$. This gives a separable Hilbert space $(\mathcal{H}, \|\ \|)$. Taking an orthonormal basis, $\{e_k\}_{k\geq 1}$, for $\mathcal{H}$, the covariance can be written $\mathbf{C}^{ij}(x-y) = \sum_k e_k^i(x) e_k^j(y)$. We will use the notation ∂_{e_k} to denote the directional derivative in the direction e_k. The covariance induces a Brownian motion on vector fields, $W_t(x) = (W_t^1(x), W_t^2(x), \ldots, W_t^d(x))$, which is centered and satisfies

$$E[W_t^i(x) W_s^j(y)] = \mathbf{C}^{ij}(x-y)\, t \wedge s, \quad t, s \geq 0, x, y \in \mathbf{R^d}.$$

This vector field may be expressed

$$W_t^i(x) = \sum_k e_k^i(x) B_t^k,$$

where the field $\{B^k : k \in \{0, 1, 2, \ldots .\}\}$ is composed of iid standard, one-dimensional Brownian motions defined on a probability space $(\Omega, \mathcal{F}, P)$.

Associated to the covariance $\mathbf{C}$ and the Brownian field $\{W_t(x) : t \geq 0, x \in \mathbf{R^d}\}$ is a flow of transition kernels induced by Markovian operators acting on $\mathbf{L^2}(\mathbf{R^d})$, $S_{s,t}$, which satisfy, according to [7],

- cocycle property:
  For all $s < u < t$ and $x \in \mathbf{R^d}$, $P - a.s.$, $S_{s,t} = S_{u,t} \circ S_{s,u}$.
- independent increments:
  For all $t_1 < t_2 < \cdots < t_n$ the kernels $\{S_{t_i,t_{i+1}} : 1 \leq i \leq n-1\}$ are independent.
- stationarity in time:
  For all $s \leq t$, the law of $S_{s,t}$ depends only on $t - s$.
- solves an SDE:
  For $f \in \mathcal{D}(\Delta)$,

$$\begin{aligned} \mathrm{d}S_{s,t} f &= \sum_k S_{s,t}(\partial_{e_k} f)\mathrm{d}B_s^k + S_{s,t}(\Delta f)\mathrm{d}s, \quad s < t, \\ S_{s,s} f &= f. \end{aligned} \tag{5}$$

Another useful formulation for the flow of transition kernels is by means of the Clark–Ocone formula. Let P_t denote the semigroup generated by the standard heat kernel on $\mathbf{R^d}$ and $f \in$

$\mathbf{L^2}(\mathbf{R^d})$. The Clark–Ocone formula satisfied by $S_{0,t}f$ reads

$$S_{0,t}f = P_t f + \sum_k \int_0^t S_{0,s}(\partial_{e_k} P_{t-s} f)\mathrm{d}B_s^k. \tag{6}$$

Our proof of the $a.s.$ Central Limit Theorem rests on the realization that the "normal" part of the distribution already appears in the Clark–Ocone formula in the form of $P_t f$ and that the noise term $\sum_k \int_0^t S_{0.s}(\partial_{e_k} P_{t-s} f)\mathrm{d}B_s^k$ tends to 0 after the usual Central Limit Theorem scaling by $\sqrt{t}$.

Finally, set $P_t^{(2)} = E[S_{0,t}^2]$, which gives the semigroup for the two point motion obtained by 'sampling' two points from the Markov transition kernel $S_{0,t}$. This is a Markov semigroup with infinitesimal generator on C^2 functions given by

$$L = \frac{1}{2}\Delta_x + \frac{1}{2}\Delta_y + \sum_{i,j} \mathbf{C}^{ij}(x-y)\partial_{x_i}\partial_{y_j}.$$

Denote the two point motion by (X_t, Y_t). The two point motion induces a distance process $d_t = \|X_t - Y_t\|$ which, due to isotropy, is a diffusion. We will use Π to denote the semigroup of the distance process;

$$\Pi_t f(d) = E_d[f(d_t)].$$

In order to get a flow of nontrivial kernels, we have to assume that 0 is a regular entrance boundary for the distance process and that this process is not absorbed at 0. This is referred to as the "diffusive without hitting" case in [7]. To be precise, this is the property that $P_0(d_t \neq 0, \forall t > 0) = 1$. The infinitesimal generator of the semigroup of the distance process, Π_t, restricted to C^2 functions can be written

$$A = \sigma^2(r)\partial_r^2 + b(r)\partial_r \tag{7}$$

with

$$\sigma^2(r) = 1 - B_L(r) \tag{8}$$

and

$$b(r) = \frac{d-1}{r}\left(1 - B_N(r)\right). \tag{9}$$

We shall be interested in a subclass of the isotropic "diffusive without hitting flows" called the Sobolev flows. These are obtained as follows (see [7].) Given constants $b > 0, m > 0$ and $0 < \alpha < 2$, set

$$F(\mathrm{d}\rho) = \frac{b\rho^{d-1}}{(d-1)(\rho^2 + m^2)^{\frac{d+\alpha}{2}}}\mathrm{d}\rho.$$

This choice of F results in a covariance $\mathbf{C}$ defined as in (2) with Fourier transform given by

$$\hat{\mathbf{C}}^{ij}(k) = c(\|k\|^2 + m^2)^{-\frac{d+\alpha}{2}} \frac{b}{d-1}\left(\delta^{ij} - \frac{k^i k^j}{\|k\|^2}\right). \tag{10}$$

We are restricting our attention to the measure preserving case as our results on the $a.s.$ Central Limit Theorem require the measure preserving assumption. Citing the results in [7], we will be interested in the cases $d \geq 3$, which guarantees that the flow of kernels is diffusive without hitting. It also gives sufficient decay in the correlator $\mathbf{C}$ to make the proof work. Notice that $B = B_L(0) = B_N(0) = \frac{b}{d} F(\mathbf{R}^+)$ and recall that we normalize so that $B = 1$ for simplicity. It was shown in [7] that there is an α_1 such that

$$\int_0^\infty \int_{S^d} \cos(r\rho u_1)\omega(\mathrm{d}u)F(\mathrm{d}\rho) = F(\mathbf{R}^+) - \alpha_1 r^\alpha + o(r^\alpha), \quad r \to 0. \tag{11}$$

Moreover, the following asymptotic formulas hold:

$$\begin{aligned} \sigma^2(r) &= \frac{b\alpha_1}{d+\alpha}(\alpha + 1 - \alpha\eta) r^\alpha (1 + o(1)), \\ b(r) &= \frac{b\alpha_1}{d+\alpha}(d - 1 + \alpha\eta) r^{\alpha-1}(1 + o(1)). \end{aligned} \tag{12}$$

Let us denote

$$\nu_t(\mathrm{d}y) = S_{0,t}(0, dy),$$

which is the (random) distribution at time t under the action of the flow of an initial point mass at the origin at time 0. Our main interest is in the behavior of the measure ν_t as $t \to \infty$. One expects, and indeed it is true, that the limit, under proper scaling, should asymptotically be a normal distribution on $\mathbf{R^d}$ for almost every realization of the background noise $\{B^k : k \in \{0, 1, 2, \ldots .\}\}$.

Theorem 1. *Let $h_t(x) = \frac{x}{\sqrt{t}}$. Then the weak convergence*

$$\nu_t \circ h_t^{-1} \to \gamma^d$$

holds a.s. where γ^d is the standard normal distribution on $\mathbf{R^d}$.

Before giving the proof of Theorem 1, we need a lemma on the decay rate of the covariance C. We denote $\|\mathbf{C}(x)\| = \max |\mathbf{C_{i,j}}(x)|$.

Lemma 2. *For any $d \geq 3$ there is a positive constant c_d such that*

$$\|\mathbf{C}(x)\| \leq c_d |x|^{-1}, \quad |x| \to \infty.$$

Proof. For $d > 4$, we can use Fourier methods. Let $f(r) = r^{-1}(r + m^2)^{-\frac{d+\alpha}{2}}$ so that $f'(r) = -r^{-2}(r + m^2)^{-\frac{d+\alpha}{2}} - \frac{d+\alpha}{2} r^{-1}(r + m^2)^{-\frac{d+\alpha+2}{2}}$. We can then write

$$\hat{\mathbf{C}}^{ij}(k) = f(\|k\|^2)\left(\delta^{ij}\|k\|^2 - k_i k_j\right)$$

and

$$\begin{aligned} \frac{\partial}{\partial k_l}\hat{\mathbf{C}}^{ij}(k) = &-\left(\|k\|^{-4}(\|k\|^2 + m^2)^{-\frac{d+\alpha}{2}} + \frac{d+\alpha}{2}\|k\|^{-2}(\|k\|^2 + m^2)^{-\frac{d+\alpha+2}{2}}\right) \\ &\times \left(\delta^{ij}\|k\|^2 - k_i k_j\right) + \|k\|^{-2}(\|k\|^2 + m^2)^{-\frac{d+\alpha}{2}}\left(2k_l\delta^{ij} - \delta^{il}k_j - \delta^{jl}k_i\right). \end{aligned}$$

From this it follows that

$$\left|\frac{\partial}{\partial k_l}\hat{\mathbf{C}}^{ij}(k)\right| \leq c\|k\|^{-(d+\alpha+1)}, \quad \|k\| \geq 1 \tag{13}$$

and that

$$\left|\frac{\partial}{\partial k_l}\hat{\mathbf{C}}^{ij}(k)\right| \leq c_1\|k\|^{-4}|k_i k_j| + c_2\|k\|^{-2}, \quad \|k\| \leq 1 \tag{14}$$

By (13) and (14) for $d \geq 5$, one obtains by Plancherel and (10),

$$\int_{\mathbf{R^d}} |x|^2 \mathbf{C}^{ij}(x)^2 \mathrm{d}x = \int_{\mathbf{R^d}} \left(\sum_{l=1}^{d} \frac{\partial}{\partial k_l}\hat{\mathbf{C}}^{ij}(k)\right)^2 \mathrm{d}k$$
$$< \infty. \tag{15}$$

Since $\mathbf{C}^{ij}(x)$ is a decreasing function of $\|x\|$, we obtain that

$$\sum_{n=1}^{\infty} n^2 (\mathbf{C}^{ij})^2(n) < \infty.$$

This implies that for some constant c_0,

$$\sup_{n\geq 1} n^2 (\mathbf{C}^{ij})^2(n) \leq c_0.$$

But then, for $n \leq x \leq n+1$,

$$(\mathbf{C}^{ij})^2(x) \leq (\mathbf{C}^{ij})^2(n) \leq \frac{c_0}{n^2} \leq \frac{c_0}{x^2}\frac{(n+1)^2}{n^2} \leq \frac{4c_0}{x^2}$$

and the proof is complete for $d > 4$. This Fourier analytic proof does not work for $d = 3$ or $d = 4$ so we resort to explicit estimates involving Bessel functions. The functions B_L and B_N are given by

$$B_L(r) = \int_0^{\infty}\int_{S^{d-1}} \cos(r\rho u_1)(1-u_1^2)\omega(\mathrm{d}u)F(\mathrm{d}\rho)$$
$$= c_{d,1} r^{-\frac{d}{2}} \int_0^{\infty} \rho^{\frac{d}{2}-1} J_{\frac{d}{2}}(r\rho)\left(\rho^2+m^2\right)^{\frac{d+\alpha}{2}} \mathrm{d}\rho \tag{16}$$

and

$$B_N(r) = \int_0^{\infty}\int_{S^{d-1}} \cos(r\rho u_1)(1-u_2^2)\omega(\mathrm{d}u)F(\mathrm{d}\rho)$$
$$= c_{d,2} r^{-\frac{d-2}{2}} \int_0^{\infty} \rho^{-\frac{d-2}{2}} J_{\frac{d-2}{2}}(r\rho)F(\mathrm{d}\rho) + c_{d,3} r^{-\frac{d}{2}} \int_0^{\infty} \rho^{-\frac{d}{2}} J_{\frac{d}{2}}(r\rho)F(\mathrm{d}\rho)$$
$$= c_{d,2} r^{-\frac{d-2}{2}} \int_0^{\infty} \rho^{\frac{d}{2}} J_{\frac{d-2}{2}}(r\rho)\left(\rho^2+m^2\right)^{\frac{d+\alpha}{2}} \mathrm{d}\rho$$
$$+ c_{d,3} r^{-\frac{d}{2}} \int_0^{\infty} \rho^{\frac{d}{2}-1} J_{\frac{d}{2}}(r\rho)\left(\rho^2+m^2\right)^{\frac{d+\alpha}{2}} \mathrm{d}\rho. \tag{17}$$

For dimension $d = 3$, the Bessel functions appearing above are (see Watson [9] page 54 where formulas are given for all half-integer indexed Bessel functions)

$$J_{\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin z$$

and

$$J_{\frac{3}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left[\frac{\sin z}{z} - \cos z\right]$$

Substituting in (16) and (17) gives in the case $d = 3$, and using the bound

$$|J_{\frac{3}{2}}(z)| \vee |J_{\frac{1}{2}}(z)| \leq cz^{-1/2},$$

$$\begin{aligned} B_L(r) &\leq c_{3,1} r^{-2} \int_0^\infty \rho^{\frac{1}{2}} \left(\rho^2 + m^2\right)^{-\frac{d+\alpha}{2}} \mathrm{d}\rho \\ &\leq C_{3,\alpha,m} r^{-2}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} B_N(r) \leq{}& c_{3,2} r^{-\frac{1}{2}} \int_0^\infty \rho^{\frac{3}{2}} \left(\frac{2}{\pi r \rho}\right)^{\frac{1}{2}} \left(\rho^2 + m^2\right)^{-\frac{d+\alpha}{2}} \mathrm{d}\rho \\ &+ c_{3,3} r^{-\frac{3}{2}} \int_0^\infty \rho^{\frac{1}{2}} \left(\frac{2}{\pi r \rho}\right)^{\frac{1}{2}} \left(\rho^2 + m^2\right)^{-\frac{d+\alpha}{2}} \mathrm{d}\rho \\ \leq{}& C_{3,\alpha,m} r^{-1} + C'_{3,\alpha,m} r^{-2}. \end{aligned} \tag{19}$$

This gives the result in $d = 3$. An entirely analogous proof works in $d = 4$. □

We now give the proof of Theorem 1.

Proof. Take $f \in C_c^2(\mathbf{R}^\mathbf{d})$ and set $f_t(x) = f(\frac{x}{\sqrt{t}})$. We note the easy fact that $P_t f_t(0) = \gamma^d(f)$ so we need to prove $\lim_{t\to\infty} \nu_t(f_t) = P_t f_t(0), a.s.$. By Ocone's formula

$$\begin{aligned} &E\left[(\nu_t(f_t) - P_t f_t(0))^2\right] \\ &\quad = E\left[\sum_{i,j} \iiint_0^t \mathbf{C}^{ij}(x-y)\partial_{x_i} P_{t-s} f_t(x) \partial_{y_j} P_{t-s} f_t(y) \mathrm{d}s \nu_s(\mathrm{d}x)\nu_s(\mathrm{d}y)\right] \\ &\quad = \iiint_0^t P_s^{(2)}((0,0),(z_1,z_2))\mathbf{C}^{ij}(z_1 - z_2)\partial_{x_i} P_{t-s} f_t(z_1)\partial_{x_j} P_{t-s} f_t(z_2) \mathrm{d}s\mathrm{d}z_1\mathrm{d}z_2 \\ &\quad \leq \frac{K}{t}\|\nabla f\|_\infty^2 \int_0^t \int \|\mathbf{C}\|(z)\Pi_s(\mathrm{d}z)\mathrm{d}s, \end{aligned} \tag{20}$$

where $\|\nabla f\|_\infty = \sup_i \sup_x |\frac{\partial f}{\partial x_i}(x)|$. In order to use a comparison theorem to dominate Π_s we perform a change of scale for the distance process of the two point motion by setting $\phi(x) = \int_0^x \frac{1}{\sigma(z)} \mathrm{d}z$ where $\sigma(x)$ is defined at (12). The generator of the process $y_t = \phi(d_t)$ is given by

$$\mathcal{L} f(y) = f''(y) + \beta(y) f'(y), \tag{21}$$

with

$$\beta(y) = \left(\frac{b}{\sigma} - \sigma'\right)(\phi^{-1}(y)).$$

One notes that $\phi(r) \sim c_L^{-\frac{1}{2}} r^{1-\frac{\alpha}{2}}$, $r \to 0$ where the constant

$$c_L = \frac{b\alpha_1}{d+\alpha}(\alpha + 1 - \alpha\eta).$$

On the other hand, B_L goes to zero so that $\phi(r) \sim r$, $r \to \infty$. Also, for small r

$$\left(\frac{b}{\sigma} - \sigma'\right)(r) \sim \frac{c_N - \frac{\alpha}{2} c_L}{\sqrt{c_L}} r^{\frac{\alpha}{2}-1}. \tag{22}$$

This implies that

$$\beta(y) \sim \frac{c}{y}, \quad \text{as } y \to 0,$$

with some positive constant c. We introduce a lemma which shows that the y diffusion has a positive drift.

Lemma 3. *For all $r > 0$,*

$$\beta(r) > 0.$$

Proof. By definition,

$$\sigma(r)\beta(r) = b(r) - \sigma(r)\sigma'(r) = \frac{d-1}{r}(1 - B_N(r)) + \frac{1}{2}B_L'(r)$$

and $\sigma(r) > 0$. Moreover, using the identity from [6]

$$(d-1)\int_{S^{d-1}} \cos(xu_1) u_2^2 \omega(\mathrm{d}u) = \int_{S^{d-1}} \cos(xu_1)(1 - u_1^2)\omega(\mathrm{d}u)$$

it follows that

$$\begin{aligned}
b(r) - \sigma(r)\sigma'(r) &= \frac{d-1}{r}(1 - B_N(r)) + \frac{1}{2}B_L'(r) \\
&= \frac{d-1}{r}\int_0^\infty \int_{S^{d-1}} (1 - \cos(r\rho u_1))\left(1 - u_2^2\right)\omega(\mathrm{d}u)F_N(\mathrm{d}\rho) \\
&\quad - \frac{1}{2}\int_0^\infty \int_{S^{d-1}} \rho u_1 \sin(r\rho u_1)(1 - u_1^2)\omega(\mathrm{d}u)F_N(\mathrm{d}\rho) \\
&= \frac{1}{r}\int_0^\infty \int_{S^{d-1}} (1 - \cos(r\rho u_1))\left(d - 1 - (1 - u_1^2)\right)\omega(\mathrm{d}u)F_N(\mathrm{d}\rho) \\
&\quad - \frac{1}{2}\int_0^\infty \int_{S^{d-1}} \rho u_1 \sin(r\rho u_1)\left(1 - u_1^2\right)\omega(\mathrm{d}u)F_N(\mathrm{d}\rho) \\
&= \frac{1}{r}\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos(r\rho\sin\theta))\left(d - 1 - \cos^2\theta\right)\cos^{d-2}\theta \mathrm{d}\theta F_N(\mathrm{d}\rho) \\
&\quad - \frac{1}{r}\int_0^\infty \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\rho\sin(r\rho\sin\theta)\cos^d\theta\sin\theta\, \mathrm{d}\theta F_N(\mathrm{d}\rho).
\end{aligned} \tag{23}$$

But integrating by parts and using

$$\sin(r\rho\sin\theta)\cos(\theta) = \frac{1}{r\rho}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(1-\cos(r\rho\sin\theta)\right),$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\cos^{d-1}\theta\sin\theta\right) &= \cos^d\theta-(d-1)\sin^2\theta\cos^{d-2}\theta \\ &= d\cos^d\theta-(d-1)\cos^{d-2}\theta\end{aligned} \tag{24}$$

we get

$$\begin{aligned}\beta(r) &= \frac{1}{r}\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-\cos(r\rho\sin\theta)\right) \\ &\quad\times\left(d-1-\frac{d-1}{2}+\left(\frac{d}{2}-1\right)\cos^2\theta\right)\cos^{d-2}\theta\,\mathrm{d}\theta\,F_N(\mathrm{d}\rho) \\ &= \frac{1}{r}\int_0^\infty\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1-\cos(r\rho\sin\theta)\right)\left(\frac{d-1}{2}+\frac{d-2}{2}\cos^2\theta\right)\cos^{d-2}\theta\,\mathrm{d}\theta\,F_N(\mathrm{d}\rho) \\ &> 0,\end{aligned}$$

as desired. □

Remark One can show that when $d=2$,

$$\beta(r) = \frac{1}{2r}\int_0^\infty(1-J_0(\rho r))F_N(\mathrm{d}\rho)$$

where J_0 is the Bessel function of order 0.

Returning now to the proof of Theorem 1, we note that $\phi(r)\sim cr^{1-\frac{\alpha}{2}}$ and that $\phi^{-1}(r)\sim cr^{1+\frac{\alpha}{2-\alpha}}$. Thus, by Lemma 2 we have for some positive constant c that

$$\|\mathbf{C}(\phi^{-1}(y))\| \le \frac{c}{y^{1+\frac{\alpha}{2-\alpha}}}\wedge 1 \equiv K(y)$$

which is a decreasing function of y. By a standard comparison argument we have by Lemma 3 that $y_t \ge |b_t|$ where b_t is a one-dimensional Brownian motion. Thus, for any nonnegative, decreasing function f, $\Pi_s f\circ\phi(x) \le P_s f(x)$, where P_s is the one-dimensional Brownian semigroup. Consequently,

$$\begin{aligned}\int_1^t \Pi_s\|\mathbf{C}\|\mathrm{d}s &\le \int_1^t P_s\|K\|\mathrm{d}s \\ &\le c\int_1^t\left(\frac{1}{\sqrt{s}}\int_{\rho\ge 1}\frac{1}{\rho^{1+\frac{\alpha}{2-\alpha}}}\mathrm{e}^{-\frac{\rho^2}{2s}}\mathrm{d}\rho+\int_{\rho\le 1}\frac{1}{\sqrt{s}}\mathrm{e}^{-\frac{\rho^2}{2s}}\mathrm{d}\rho\right)\mathrm{d}s \\ &\le k_1\int_1^t s^{-(\frac{1}{2}+\frac{\alpha}{2(2-\alpha)})}\mathrm{d}s+k_2 \\ &\le k_1 t^{\frac{1}{2}-\frac{\alpha}{2(2-\alpha)}}+k_2.\end{aligned} \tag{25}$$

Similarly,

$$\int_r^t \Pi_s\|\mathbf{C}\|\mathrm{d}s \le k_1\left(t^{\frac{1}{2}-\frac{\alpha}{2(2-\alpha)}}-r^{\frac{1}{2}-\frac{\alpha}{2(2-\alpha)}}\right)+k_2. \tag{26}$$

Consider now the sequence $t_n = n^3$. Then we see from (20) and (25) that

$$\sum_n E\left[\left(\nu_{t_n}(f_{t_n}) - \gamma^d(f)\right)^2\right] < \infty$$

and so

$$\lim_{n\to\infty} \nu_{t_n}(f_{t_n}) = \gamma^d(f), \quad a.s. \tag{27}$$

Now, for the purpose of controlling the oscillations between n^3 and $(n+1)^3$, set

$$\epsilon_n = \sup_{t_n \le t \le t_{n+1}} \left| \int_{t_n}^t \mathrm{d}\nu_s(f_s) \right|.$$

By (41),

$$\begin{aligned}
\int_{t_n}^t \mathrm{d}\nu_s f_s &= \int_{t_n}^t \sum_k S_{0,s}(\partial_{e_k} f_s)\mathrm{d}B_s^k + \int_{t_n}^t \nu_s\left(\frac{\partial}{\partial s}(f_s)\right)\mathrm{d}s + \int_{t_n}^t \nu_s(\Delta f_s)\mathrm{d}s \\
&= \int_{t_n}^t \sum_k S_{0,s}\left\langle \nabla f_s,\, e_k \mathrm{d}B_s^k \right\rangle - \frac{1}{2}\int_{t_n}^t \nu_s\left(\frac{1}{s^{3/2}}\langle (\nabla f)_s,\, y\rangle\right)\mathrm{d}s + \int_{t_n}^t \nu_s(\Delta f_s)\mathrm{d}s \\
&= \int_{t_n}^t \frac{1}{\sqrt{s}}\nu_s(\langle (\nabla f)_s, \mathrm{d}W_s\rangle) - \frac{1}{2}\int_{t_n}^t \int \langle \nabla f(y), y\rangle_s \frac{1}{s}\nu_s(\mathrm{d}y)\mathrm{d}s + \int_{t_n}^t \frac{1}{s}\nu_s((\Delta f)_s)\mathrm{d}s
\end{aligned}$$

This implies that

$$\epsilon_n \le A_n + B_n + C_n$$

where

$$A_n = \sup_{t_n \le t \le t_{n+1}} \int_{t_n}^t \int |\langle \nabla f(y), y\rangle_s| \frac{1}{s}\nu_s(\mathrm{d}y)\mathrm{d}s, \tag{28}$$

$$B_n = \sup_{t_n \le t \le t_{n+1}} \left| \int_{t_n}^t \frac{1}{s}\nu_s((\Delta f)_s)\mathrm{d}s \right| \tag{29}$$

and

$$C_n = \sup_{t_n \le t \le t_{n+1}} \left| \int_{t_n}^t \frac{1}{\sqrt{s}}\nu_s(\langle (\nabla f)_s, \mathrm{d}W_s\rangle) \right|. \tag{30}$$

Since $f \in C_c^2$, there is a constant M such that for all $y \in \mathbf{R^d}$,

$$|\langle \nabla f(y), y\rangle| \le M. \tag{31}$$

Also, by our choice of t_n,

$$\lim_{n\to\infty} \frac{t_{n+1} - t_n}{t_n} = 0, \tag{32}$$

Thus, from (28), (31) and (32), we have

$$\lim_{n\to\infty} A_n = 0. \tag{33}$$

Similarly, since $f \in C_c^2$, there is a constant M such that

$$|\Delta f(y)| \le M, \quad y \in \mathbf{R^d}$$

and so

$$B_n \leq M \frac{t_{n+1} - t_n}{t_n} \tag{34}$$

whence

$$\lim_{n \to \infty} B_n = 0. \tag{35}$$

Finally, by Doob's inequality,

$$\begin{aligned} E\left[C_n^2\right] &\leq 4E\left[\left(\int_{t_n}^{t_{n+1}} \frac{1}{\sqrt{s}} \nu_s(\langle(\nabla f)_s, \mathrm{d}W_s\rangle)\right)^2\right] \\ &\leq \frac{4}{t_n} E\left[\iiint_{t_n}^{t_{n+1}} \nu_s(\mathrm{d}x)\nu_s(\mathrm{d}y)\, \langle(\nabla f)_s(x), (\nabla f)_s(y)\mathbf{C}(x-y)\rangle\, \mathrm{d}s\right] \\ &\leq \frac{4}{t_n} \|\nabla f\|_\infty^2 \iiint_{t_n}^{t_{n+1}} P_s^{\otimes 2}(z_1, z_2)\|C(z_1 - z_2)\|\mathrm{d}z_1 \mathrm{d}z_2 \end{aligned}$$

We will thus have established $a.s.$ convergence once we have shown

$$\sum_{n \geq 1} \frac{1}{t_n} \int_{t_n}^{t_{n+1}} \Pi_s \|\mathbf{C}\| \mathrm{d}s < \infty.$$

Using (26),

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{t_n} \int_{t_n}^{t_{n+1}} \Pi_s \|\mathbf{C}\| \mathrm{d}s &\leq \sum_{n \geq 1} n^{-3} \left(k_1((n+1)^{\frac{3}{2} + \frac{3\epsilon}{2}} - n^{\frac{3}{2} + \frac{3\epsilon}{2}}) + k_2\right) \\ &\leq k_3 \sum_{n \geq 1} n^{-\frac{3}{2} + \frac{3\epsilon}{2}} \\ &< \infty. \end{aligned} \tag{36}$$

The proof is complete. □

3. Isotropic flows

We now turn our attention to the standard, Lebesgue measure preserving, isotropic Brownian flows studied in [6] and [1]. In contrast to the Kraichnan flows dealt with previously, these flows are generated by Brownian vector fields with smoother correlations. For the standard isotropic Brownian flows, we assume we are given a measures F with finite moments of order up to two, $\int_0^\infty \rho^2 F(\mathrm{d}\rho) < \infty$ and such that $F(\{0\}) = 0$. As at (3) and (4) we define for $r > 0$, and for ω the normalized Lebesgue measure on the unit sphere $\mathbf{S^{d-1}}$ in $\mathbf{R^d}$,

$$\begin{aligned} B_L(r) &= \int_0^\infty \int_{S^{d-1}} \cos(r\rho u_1)(1 - u_1^2) F(\mathrm{d}\rho), \\ B_N(r) &= \int_0^\infty \int_{S^{d-1}} \cos(r\rho u_1)(1 - u_2^2) F(\mathrm{d}\rho). \end{aligned} \tag{37}$$

Then the general isotropic covariance associated to measure preserving flows has the form

$$\mathbf{C}^{ij}(z) = \delta^{ij} B_N(\|z\|) + \frac{z^i z^j}{\|z\|^2}(B_L(\|z\|) - B_N(\|z\|)) \tag{38}$$

we consider a Brownian vector field $\{W_t(x) : x \in \mathbf{R^d}\}$ with correlations given by

$$E[W_t^i(x)W_s^j(y)] = \mathbf{C}^{ij}(x-y)\, t \wedge s. \tag{39}$$

Then, in contrast to the case of Kraichnan flows, we get a flow of diffeomorphisms by means of strong solutions of the SDE

$$\Phi_{s,t}(x) = x + \int_s^t \mathrm{d}W_u(\Phi_{s,u}(x)). \tag{40}$$

To connect with the notation of the Kraichnan flows we note that the flow of transition kernels is now given by

$$S_{s,t} f(x) = f(\Phi_{s,t}(x)).$$

Once again we have all of the previous properties,

- cocycle property:
  For all $s < u < t$ and $x \in \mathbf{R^d}$, $P - a.s.$, $S_{s,t} = S_{u,t} \circ S_{s,u}$.
- independent increments:
  For all $t_1 < t_2 < \cdots < t_n$ the kernels $\{S_{t_i,t_{i+1}} : 1 \le i \le n-1\}$ are independent.
- stationarity in time:
  For all $s \le t$, the law of $S_{s,t}$ depends only on $t - s$.
- solves an SDE:
  For $f \in \mathcal{D}(\Delta)$,

$$\begin{aligned} \mathrm{d}S_{s,t} f &= \sum_k S_{s,t}(\partial_{e_k} f)\mathrm{d}B_s^k + S_{s,t}(\Delta f)\mathrm{d}s, \quad s < t, \\ S_{s,s} f &= f. \end{aligned} \tag{41}$$

In addition, the Clark–Ocone formula still holds

$$S_{0,t} f = P_t f + \sum_k \int_0^t S_{0,s}(\partial_{e_k} P_{t-s} f)\mathrm{d}B_s^k. \tag{42}$$

The flow $\Phi_{s,t}$ has Lyapunov exponents,

$$\lambda_i = \frac{d - 2i + 1}{2(d+2)} \int_0^\infty \rho^2 F(\mathrm{d}\rho), \quad i = 1, 2, \ldots, d. \tag{43}$$

We will need an estimate of the decay rate of $\mathbf{C}$ at infinity. For this we use that

$$\begin{aligned} &\int_{S^{d-1}} \cos(xu_1)\omega(\mathrm{d}u) = 2^{\frac{d-2}{2}} \Gamma\left(\frac{d}{2}\right) x^{\frac{2-d}{2}} J_{\frac{d-2}{2}}(x) \\ &(d-1)\int_{S^{d-1}} \cos(xu_1)u_2^2\omega(\mathrm{d}u) = \int_{S^{d-1}} \cos(xu_1)\omega(\mathrm{d}u)(1-u_1^2)\omega(\mathrm{d}u) \\ &\qquad = 2^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \frac{d-1}{2} x^{-\frac{d}{2}} J_{\frac{d}{2}}(x) \end{aligned} \tag{44}$$

Then by the definition of B_L and B_N at (37) this gives

$$\begin{aligned} B_L(r) &= 2^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \frac{d-1}{2} \int_0^\infty (r\rho)^{-\frac{d}{2}} J_{\frac{d}{2}}(r\rho) F(\mathrm{d}\rho), \\ B_N(r) &= 2^{\frac{d-2}{2}} \Gamma\left(\frac{d}{2}\right) \int_0^\infty (r\rho)^{-\frac{d}{2}} \left[r\rho J_{\frac{d-2}{2}}(r\rho) - J_{\frac{d}{2}}(r\rho) \right] F(\mathrm{d}\rho). \end{aligned} \tag{45}$$

Using the bounds from (5.16.1) in [5]

$$\begin{aligned} J_\nu(x) &\sim \frac{x^\nu}{2^\nu \Gamma(1+\nu)}, \quad x \to 0, \\ J_\nu(x) &\sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{1}{2}\nu\pi - \frac{1}{4}\pi\right), \quad x \to \infty, \end{aligned} \tag{46}$$

it follows readily that

$$B_N(r) = O\left(r^{\frac{1-d}{2}}\right), \qquad B_L(r) = O\left(r^{\frac{1-d}{2}}\right), \quad r \to \infty. \tag{47}$$

From (38) and (47) we conclude that Lemma 2 holds for $\mathbf{C}$ as given in (38). That is,

$$|\mathbf{C}(x)| \le c_d |x|^{-1}, \quad d \ge 3. \tag{48}$$

We also note that

$$B_L'(r) = -2^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \frac{d-1}{2} \int_0^\infty (r\rho)^{-\frac{d}{2}} J_{\frac{d}{2}+1}(r\rho) \rho F(\mathrm{d}\rho),$$

which gives

$$B_L'(r) = O(r^{\frac{1-d}{2}}), \quad r \to \infty. \tag{49}$$

Let us now examine the case $r \to 0$. Here we have, recalling the normalization $B_L(0) = B_N(0) = 1$,

$$\begin{aligned} 1 - B_L(r) &= (\beta + \gamma) r^2 + O(r^4), \\ 1 - B_N(r) &= \beta r^2 + O(r^4), \\ B_L'(r) &= -2(\beta+\gamma) r + O(r^3), \quad r \to 0 \end{aligned} \tag{50}$$

where

$$\begin{aligned} 2\beta &= \frac{d+1}{d(d+2)} \int_0^\infty \rho^2 F(\mathrm{d}\rho), \\ \gamma &= -\frac{1}{d(d+2)} \int_0^\infty \rho^2 F(\mathrm{d}\rho). \end{aligned} \tag{51}$$

Note that

$$2(\beta+\gamma) = \frac{d-1}{d(d+2)} \int_0^\infty \rho^2 F(\mathrm{d}\rho). \tag{52}$$

and for $d \geq 3$,

$$\lambda_1 = (d-2)\beta - \gamma = \frac{d-1}{2(d+2)} \int_0^\infty \rho^2 F(\mathrm{d}\rho) > 0. \tag{53}$$

In this case, we have the following result.

Theorem 4. *Assume $d \geq 3$ and that ν is a probability measure on $\mathbf{R^d}$ which satisfies the energy condition:*

$$\int_{\|x-y\|<1} \|x-y\|^{-\frac{d}{2}} \nu(\mathrm{d}x)\nu(\mathrm{d}y) < \infty. \tag{54}$$

Set $\nu_t(A) = \nu(\{x : \phi_{0,t}(x) \in \sqrt{t}A\})$. Then we have the weak convergence

$$\lim_{t\to\infty} \nu_t = \gamma^d, \quad a.s. \tag{55}$$

where γ^d is the standard normal distribution on $\mathbf{R^d}$.

Proof. Take $f \in C_c^2(\mathbf{R^d})$ and set $f_t(x) = f(\frac{x}{\sqrt{t}})$. Then observe that $\int f_t \mathrm{d}\nu_t = \int S_{0,t} f_t \mathrm{d}\nu$. Next notice that since $\lim_{t\to\infty} P_t f_t(x) = \gamma^d(f)$, for all x, we have $\lim_{t\to\infty} \int P_t f_t(x)\nu(\mathrm{d}x) = \gamma^d(f)$. So to prove the theorem it suffices to show

$$\lim_{t\to\infty} \left(\int f_t \mathrm{d}\nu_t - \int P_t f_t(x)\nu(\mathrm{d}x) \right) = 0, \quad a.s.$$

For this, by the Clark–Ocone formula,

$$\begin{aligned} E\left[\left(\nu_t(f_t) - \int P_t f_t(x)\nu(\mathrm{d}x)\right)^2\right] &= E\left[\left(\int S_{0,t} f_t \mathrm{d}\nu - \int P_t f_t(x)\nu(\mathrm{d}x)\right)^2\right] \\ &= E\left[\sum_{i,j} \int_0^t \int_{\mathbf{R^{2d}}} \partial_{x_i} P_{t-s} f_t(x) \partial_{y_j} P_{t-s} f_t(y) \nu_t(\mathrm{d}x)\nu_t(\mathrm{d}y)\mathrm{d}s\right] \\ &= \int_0^t \int_{\mathbf{R^{4d}}} P^{(2)}((x,y),(z_1,z_2)) C^{ij}(z_1 - z_2) \\ &\quad \times \partial_{x_i} P_{t-s} f_t(x) \partial_{x_j} P_{t-s} f_t(z_2) \mathrm{d}z_1 \mathrm{d}z_2 \nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \\ &\leq \frac{K}{t} \|\nabla f\|_\infty^2 \int_0^t \int \|C\|(z) \Pi_s(x-y, dz)\mathrm{d}s\nu(\mathrm{d}x)\nu(\mathrm{d}y). \end{aligned} \tag{56}$$

As in the case of Kraichnan flows, the behavior of the distance process becomes more transparent after a change of scale. Recall that the generator of the distance process is given by

$$\mathcal{A}f(r) = (1 - B_L(r))\, f''(r) + \frac{d-1}{r}(1 - B_N(r))\, f'(r).$$

Next we perform a change of scale by setting $u = \phi(r)$ which gives that

$$\begin{aligned} \mathcal{A}f \circ \phi(r) &= (1 - B_L(r))\, (\phi'(r))^2 f'' \circ \phi(r) \\ &\quad + \left(\frac{d-1}{r}(1 - B_N(r))\, \phi'(r) + (1 - B_L(r))\, \phi''(r) \right) f' \circ \phi(r). \end{aligned} \tag{57}$$

Choosing

$$\phi'(r) = (2\,(1 - B_L(r)))^{-\frac{1}{2}} \tag{58}$$

we get

$$\phi''(r) = (2\,(1 - B_L(r)))^{-\frac{3}{2}}\, B_L'(r) \tag{59}$$

and the generator of the process $u_t = \phi(r_t)$ takes the form

$$\mathcal{A}f(u) = \tfrac{1}{2} f''(u) + b(u) f'(u), \tag{60}$$

where

$$b(u) = \frac{(d-1)\left(1 - B_N(\phi^{-1}(u))\right)}{\phi^{-1}(u)\sqrt{2\left(1 - B_L(\phi^{-1}(u))\right)}} + \frac{B_L'(\phi^{-1}(u))}{2\sqrt{2\left(1 - B_L(\phi^{-1}(u))\right)}}. \tag{61}$$

For the sake of using a comparison theorem we need to examine the asymptotics of the drift coefficient of the diffusion u_t as $u \to -\infty$ and as $u \to \infty$. First observe that

$$\phi(r) = \int_1^r \frac{1}{\sqrt{2\,(1 - B_L(s))}} \mathrm{d}s. \tag{62}$$

In view of (47) and (50) it holds that

$$\phi(r) \sim \begin{cases} \sqrt{2(\beta+\gamma)}^{-1} \ln r, & r \to 0, \\ \dfrac{r}{\sqrt{2}}, & r \to \infty. \end{cases} \tag{63}$$

This implies

$$\phi^{-1}(u) \sim \begin{cases} \mathrm{e}^{\sqrt{2(\beta+\gamma)}u}, & u \to -\infty, \\ \sqrt{2}u, & u \to \infty. \end{cases} \tag{64}$$

Consequently, for some $p \in (0, 1)$ with p near 1, there is a u_p such that

$$b(u) \geq p \frac{\lambda_1}{\sqrt{2(\beta+\gamma)}}, \quad u \leq u_p \tag{65}$$

and

$$b(u) \geq 0, \quad u \geq u_p. \tag{66}$$

By a translation of ϕ which does not affect the above asymptotics, we may assume that $u_p = 0$ and write $c_0 = p\frac{\lambda_1}{\sqrt{2(\beta+\gamma)}}$. Thus, we define the process

$$v_t = v_0 + b_t + c_0 \int_0^t 1_{\{v_s < 0\}} \mathrm{d}s, \quad v_0 = u_0 = \phi(d_0). \tag{67}$$

Then by elementary comparison argument,

$$v_t \leq u_t. \tag{68}$$

Thus, by (48), we conclude

$$\begin{aligned} C(d_t) &= C(\phi^{-1}(u_t)) \\ &\le C(\phi^{-1}(v_t)) \\ &\le c\left(1 \wedge \frac{1}{\phi^{-1}(v_t)}\right) \\ &\le c\left(1 \wedge \frac{1}{v_t}\right) \\ &\le c\left(1_{\{v_t<1\}} + \frac{1}{v_t}1_{\{v_t\ge 1\}}\right), \end{aligned} \tag{69}$$

where the value of c may change from line to line. Finally, define the stopping time

$$T_0 = \inf\{s > 0 : v_s = 0\}.$$

The expression in the last line of (56) is bounded from above as follows (where we simplify the notation by writing $a = \phi(\|x - y\|)$:

$$\begin{aligned} &\frac{K}{t}\int_0^t \int \|C\|(z)\Pi_s(x-y, dz)\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \\ &\quad \le \frac{K}{t}\int_0^t \int E_a\left[\|C\|(\phi^{-1}(v_s))\right]\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \\ &\quad \le \frac{K}{t}\int_0^t \int E_a\left[1_{\{v_s<1\}} + \frac{1}{v_s}1_{\{v_s\ge 1\}}\right]\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \\ &\quad = \frac{K}{t}\int_0^t \int E_a\left[1_{\{T_0>s\}}\right]\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \\ &\qquad + \frac{K}{t}\int_0^t \int E_a\left[(1_{\{v_s<1\}} + \frac{1}{v_s}1_{\{v_s\ge 1\}})1_{\{T_0\le s\}}\right]\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s. \end{aligned} \tag{70}$$

By a simple Girsanov transformation, starting from $a < 0$, which corresponds to $\|x - y\| < 1$,

$$P_a(T_0 \in \mathrm{d}\tau) = \mathrm{e}^{-c_0 a - \frac{c_0^2\tau}{2}}\mu_a(T_0 \in \mathrm{d}\tau), \tag{71}$$

where $\mu_a(T_0 \in \mathrm{d}\tau)$ is the distribution of the hitting time to 0 of one-dimensional Brownian motion starting at a. For the first term, we can bound by

$$\begin{aligned} &\frac{K}{t}\int_0^t \int_{a<0} E_a\left[1_{\{T_0>s\}}\right]\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \\ &\quad \le \frac{K}{t}\int_0^t \int_{a<0}\int_s^\infty \mathrm{e}^{-c_0 a - \frac{c_0^2\tau}{2}}\mu_a(T_0 \in \mathrm{d}\tau)\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \\ &\quad \le \frac{K}{t}\int_0^t \int_{a<0} \mathrm{e}^{-\frac{c_0^2 s}{2}}\mathrm{e}^{-c_0 a}\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \\ &\quad \le \frac{K}{c_0^2 t}(1 - \mathrm{e}^{-\frac{c_0^2 t}{2}})\int_{a<0} \mathrm{e}^{-c_0 a}\nu(\mathrm{d}x)\nu(\mathrm{d}y) \\ &\quad \le \frac{K}{c_0^2 t}\int_{\|x-y\|<\phi^{-1}(0)} \|x-y\|^{-\frac{pd}{2}}\nu(\mathrm{d}x)\nu(\mathrm{d}y) \end{aligned} \tag{72}$$

This goes to 0 as $t \to \infty$ since $\int_{\|x-y\|<\phi^{-1}(0)} \|x-y\|^{-\frac{d}{2}} \nu(\mathrm{d}x)\nu(\mathrm{d}y) < \infty$ by our assumption on ν. For the second term in (70),

$$\frac{K}{t}\int_0^t \int E_a\left[\left(1_{\{v_s<1\}} + \frac{1}{v_s}1_{\{v_s\geq 1\}}\right)1_{\{T_0\leq s\}}\right]\mathrm{d}s\,\nu(\mathrm{d}x)\nu(\mathrm{d}y)$$
$$= \frac{K}{t}\int_0^t\int_0^s\int E_0\left[1\wedge\frac{1}{|v_{s-r}|}\right]P_a(T_0\in \mathrm{d}r)\mathrm{d}r\mathrm{d}s\,\nu(\mathrm{d}x)\nu(\mathrm{d}y).$$

Denote $Q(\mathrm{d}\tau) = \int P_a(T_0 \in d\tau)\nu(\mathrm{d}x)\nu(\mathrm{d}y)$ and $\psi(u) = E_0\left[1\wedge\frac{1}{|v_u|}\right]$. Then the last term can be expressed as

$$\frac{K}{t}\int_0^t\int_0^s \psi(s-\tau)\mathrm{d}s\,Q(\mathrm{d}\tau) = \frac{K}{t}\int_0^t\left(\int_0^{t-\tau}\psi(u)\mathrm{d}u\right)Q(\mathrm{d}\tau).$$

By integrating with respect to Q, we see that with a possible new value for K

$$\frac{K}{t}\int_0^t\left(\int_0^{t-\tau}\psi(u)\mathrm{d}u\right)Q(\mathrm{d}\tau) \leq \frac{K}{t}\int_0^t\psi(u)\mathrm{d}u$$

As an aside, we remark that as $N\to\infty$, $\frac{v_{Nt}}{\sqrt{N}}$ converges in law towards a reflected Brownian motion. Indeed, it is a diffusion with generator $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + c_0\sqrt{N}1_{\{x<0\}}\frac{\mathrm{d}}{\mathrm{d}x}$. The scale function and the speed measure for this diffusion converge towards those of the reflected Brownian motion. From that observation, we deduce that $P_0(v_u < u^{\frac{1}{2}-\varepsilon})$ tends to 0 as $u\to\infty$. But we need a better estimate to conclude the proof.

For this better estimate, introduce $A_t^{\pm} = \int_0^t 1_{\{\pm v_s>0\}}\mathrm{d}s$ and $\sigma_t = \inf\{u :\ A_u^+ = t\}$. Note that $\sigma_t \geq t, a.s.$ and v_{σ_t} is a reflected Brownian motion. Moreover,

$$\begin{aligned}\int_0^t\psi(u)\mathrm{d}u &= E_0\left[\int_0^t\left(1\wedge\frac{1}{|v_s|}\right)\mathrm{d}s\right]\\ &\leq E_0\left[\int_0^{\sigma_t}\left(1\wedge\frac{1}{|v_s|}\right)\mathrm{d}s\right]\\ &= E_0\left[\int_0^{\sigma_t}1_{\{v_s<0\}}\left(1\wedge\frac{1}{|v_s|}\right)\mathrm{d}s\right] + E_0\left[\int_0^{\sigma_t}1_{\{v_s>0\}}\left(1\wedge\frac{1}{|v_s|}\right)\mathrm{d}s\right]\\ &\leq E_0\left[A_{\sigma_t}^-\right] + E_0\left[\int_0^t\left(1\wedge\frac{1}{v_{\sigma_s}}\right)\mathrm{d}s\right].\end{aligned}\tag{73}$$

For the first term, note that $v_t - c_0A_t^-$ is a square integrable martingale and that $E_0[\sigma_t]$ is finite. Therefore, $E_0[A_{\sigma_t}^-] = c_0^{-1}E_0[v_{\sigma_t}] = c_0^{-1}\sqrt{t}$.

If χ is a standard normal random variable the second term is smaller than

$$\begin{aligned}E_0\left[\int_0^t\left(1\wedge\frac{1}{v_{\sigma_s}}\right)\mathrm{d}s\right] &= \int_0^t\left(P\left(|\chi|<\frac{1}{\sqrt{s}}\right)+\frac{2}{\sqrt{s}}E\left[\frac{1}{\chi}1_{\{\chi>\frac{1}{\sqrt{s}}\}}\right]\right)\mathrm{d}s\\ &\leq c\int_0^t\left(\frac{1}{\sqrt{s}}\left(1+\int_{\frac{1}{\sqrt{s}}}^\infty\frac{1}{x}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x\right)\right)\mathrm{d}s\\ &\leq c\sqrt{t}+c\int_0^t\left(\frac{1}{\sqrt{s}}\int_{\frac{1}{\sqrt{s}}}^1\frac{1}{x}\mathrm{d}x+\frac{1}{\sqrt{s}}\int_1^\infty\frac{1}{x}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x\right)\mathrm{d}s\end{aligned}$$

$$\leq c\left(\sqrt{t} + \int_1^t \frac{\ln s}{\sqrt{s}}\mathrm{d}s\right)$$
$$\leq C\sqrt{t}\ln t.$$

Thus,

$$\frac{K}{t}\int_0^t \int \|C\|(z)\Pi_s(x-y,dz)\nu(\mathrm{d}x)\nu(\mathrm{d}y)\mathrm{d}s \leq C\frac{\ln t}{\sqrt{t}}$$

This estimate allows us to proceed as in the proof of Theorem 1 starting from (26) and prove the $a.s.$ convergence exactly in the same way as before. □

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